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Effects of plane progressive irrotational waves on thermal boundary layers

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The average changes in the structure of thermal boundary layers at the surface of bodies of water produced by various types of surface waves are computed. The waves are two-dimensional plane progressive irrotational waves of unchanging shape. They include deep-water linear waves, deep-water capillary waves of arbitrary amplitude, Stokes waves, and the deep-water gravity wave of maximum amplitude.

The results indicate that capillary waves can decrease mean temperature gradients by factors of as much as 9·0, if the average heat flux at the air–water interface is independent of the presence of the waves. Irrotational gravity waves can decrease the mean temperature gradients by factors no more than 1·381.

Of possible pedagogical interest is the simplicity of the heat conduction equation for two-dimensional steady irrotational flows in an inviscid incompressible fluid if the velocity potential and the stream function are taken to be the independent variables.

1. Introduction

A strong thermal boundary layer is usually present at the top of the ocean (see Osborne (1964) for a complete discussion, including references to early experimental and observational evidence; see McAlister & McLeish (1969) for later references and discussion). Evaporation, radiation, and heat transport establish the boundary layer, which is approximately 1–2 mm thick and across which the temperature variation is of the order of 0·5 °C. Because the layer thickness and thermal conductivity of water have known values, the time scale for formation of a layer is known to be of the order of 10 sec.

Osborne (1965) and O'Brien (1967) have shown that surface waves can significantly alter the boundary-layer structure. Osborne treated linear waves, and O'Brien treated Gerstner waves, which are not irrotational. Both types of wave are only an imperfect representation of real ocean waves, which are frequently large-amplitude waves, and are nearly irrotational. In fact, any vorticity generated by wind stresses is oppositely directed from that of a Gerstner wave travelling down-wind (see Lamb (1932, pp. 421–3) for an exposition of the Gerstner wave). In addition, recent experimental evidence (Hill 1970) indicates that capillary waves may alter the thermal boundary layer much more than is possible for the waves considered by Osborne and O'Brien.

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In this paper we consider the effects on thermal boundary layers produced by irrotational waves and include waves of finite amplitude. In § 2 the equation is derived which governs heat conduction within steady irrotational flows in incompressible inviscid fluids, assuming that thermal effects such as buoyancy do not alter the flow field. The independent variables in the flow description are the natural pair for plane progressive waves of unchanging shape: the velocity potential ϕ and the stream function ψ. The heat conduction equation in ϕ, ψ co-ordinates is so elementary for steady boundary layers that explicit solutions are easily obtained. Two relevant boundary conditions are discussed in § 2.2. Then the lowest-order effects of four kinds of progressive waves on thermal boundary layers are calculated: linearized sinusoidal waves in deep water in § 3; deep-water capillary waves of arbitrary amplitude (Crapper 1957) in § 4; Stokes waves (Stokes 1847, 1880) and deep-water gravity waves of maximum amplitude (Michell 1893 or Havelock 1918) in §§ 5 and 6, respectively. Some conclusions are set forth in § 7.

2. Basic formalism

2.1. *The heat conduction equation*

For two-dimensional problems the equation governing heat conduction in a fluid is

$$dT/dt = \kappa(\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2), \tag{2.1}$$

where T denotes the temperature, t the time, κ the thermal diffusivity

($\kappa = 1{\cdot}4 \times 10^{-3}\,\mathrm{cm^2/sec}$ for water at room temperature),

and x and y the horizontal and vertical co-ordinates respectively. In two dimensions the convective derivative is defined as

$$d/dt = \partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y, \tag{2.2}$$

where u denotes the horizontal velocity and v the vertical velocity.

Because the fluid is taken to be inviscid and the flow irrotational, the velocity may be written as the gradient of a potential function ϕ, and can be related to a stream function ψ:

$$u = \partial\phi/\partial x = \partial\psi/\partial y; \tag{2.3}$$

$$v = \partial\phi/\partial y = -\,\partial\psi/\partial x. \tag{2.4}$$

Following Stokes (1880) we choose to regard ϕ and ψ as the independent variables in the heat conduction problem. Then the convective derivative becomes

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\phi}{dt}\frac{\partial}{\partial \phi} + \frac{d\psi}{dt}\frac{\partial}{\partial \psi} \tag{2.5}$$

$$= \frac{\partial}{\partial t} + (u^2+v^2)\frac{\partial}{\partial \phi}. \tag{2.6}$$

Equation (2.6) is valid only for steady flows, where ϕ and ψ are not explicit functions of the time. It can, of course, be derived directly from (2.2) using the Cauchy–Riemann conditions equations (2.3)–(2.4).

We may also transform the Laplacian in (2.1) directly into ϕ, ψ co-ordinates. However, we choose to start with the following definitions of the two-dimensional Laplacians:

$$\nabla^2_{x,y} T \equiv \lim_{\delta A \to 0} \frac{1}{\delta A} \oint \frac{\partial T}{\partial n} ds; \tag{2.7}$$

$$\nabla^2_{\phi,\psi} T \equiv \lim_{\delta A' \to 0} \frac{1}{\delta A'} \oint \frac{\partial T}{\partial n'} ds', \tag{2.8}$$

where unprimed characters refer to x, y space, and primed characters refer to ϕ, ψ space. The path of integration in each line integral encloses an area δA or $\delta A'$. If the domains of integration are a mapping between x, y and ϕ, ψ, then the integrals are identical because such a mapping is conformal. Thus,

$$\nabla^2_{x,y} T = \frac{A'}{A} \nabla^2_{\phi,\psi} T = \frac{\partial(\phi,\psi)}{\partial(x,y)} \nabla^2_{\phi,\psi} T = (u^2+v^2)\, \nabla^2_{\phi,\psi} T. \tag{2.9}$$

Upon substitution of (2.9) and (2.6) into (2.1), the heat conduction equation takes the remarkably simple form

$$(u^2+v^2)^{-1} \frac{\partial T}{\partial t} + \frac{\partial T}{\partial \phi} = \kappa \left(\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial \psi^2} \right). \tag{2.10}$$

Equation (2.10) is in an excellent form for numerical calculation even for quite complicated velocity fields, and has the useful property that wavy domains in x, y appear as rectangular domains in ϕ, ψ. If the temperature field is steady, further simplification is possible, for then the heat conduction equation is linear, and the velocity field is absent.

$$\frac{\partial T}{\partial \phi} = \kappa \left(\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial \psi^2} \right). \tag{2.11}$$

Equation (2.11) has been derived by a different method by Boussinesq (1905) but equation (2.10) represents an original contribution. Its application is appropriate to those problems which have steady velocity fields but unsteady temperature fields; this application may be rare. The problems solved in this paper involve both steady temperature fields and sufficiently simple thermal boundary conditions that (2.11) applies. As an interesting example of a modern calculation exploiting (2.11) see Grosh & Cess (1958).

2.2. *Boundary conditions and basic solutions*

The velocity fields to be considered represent waves and are periodic in ϕ, the period being $\Delta\phi = \lambda c$ ($\lambda \equiv 2\pi/k$ is the wavelength, and c is the phase speed of the wave). Thermal boundary conditions are also taken as periodic in ϕ. This choice of boundary condition will permit solutions to (2.11) which possess no horizontal temperature gradients on any scale other than a wavelength. Then a general solution to (2.11) is

$$T = \sum_{j=0}^{\infty} [T_j^{(c)}(\psi) \cos(jk\phi/c) + T_j^{(s)}(\psi) \sin(jk\phi/c)]. \tag{2.12}$$

Because (2.11) governs (2.12), the $T_j^{(c,s)}$ satisfy

$$\partial^2 T_0^{(c)}/\partial\psi^2 = 0 \quad (j = 0), \tag{2.13}$$

$$\left.\begin{aligned} -\alpha_j T_j^{(c)} &= -\alpha_j^2 \kappa T_j^{(s)} + \kappa\, \partial^2 T_j^{(s)}/\partial\psi^2 \\ +\alpha_j T_j^{(s)} &= -\alpha_j^2 \kappa T_j^{(c)} + \kappa\, \partial^2 T_j^{(c)}/\partial\psi^2 \end{aligned}\right\} \quad (j \neq 0), \tag{2.14}$$

where $\alpha_j \equiv jk/c$. The problem of obtaining $T = T(\phi, \psi)$ is well-posed if any of the classic boundary conditions for elliptic equations is specified. Here we choose to consider that the water temperature is constant along a streamline at the bottom of the thermal boundary layer, i.e.

$$T = T_w = \text{constant} \quad (\psi = \psi_0). \tag{2.15}$$

At the upper boundary, to be thought of as the streamline marking the air–water interface, we assume either Neumann conditions (I) or Dirichlet conditions (II), i.e.

$$\text{(I)} \quad \partial T/\partial\psi = N(\phi) \quad (\psi = 0), \tag{2.16}$$

or

$$\text{(II)} \quad T = D(\phi) \quad (\psi = 0). \tag{2.17}$$

The Neumann conditions are the more natural for the air–sea interface, at least in near-neutral conditions, because the dominant processes establishing the thermal boundary layer are evaporation and radiation, both of which involve relatively-constant heat fluxes whether waves are present or not. Dirichlet conditions are included for comparison with O'Brien's Gerstner-wave results.

Let $Q(\phi) \equiv$ heat flux out of the water, and now define y as positive downward and ψ as positive in the direction away from the free surface. Then

$$Q(\phi) \equiv \kappa \frac{\partial T}{\partial n} = \kappa \frac{d\psi}{dn}\frac{\partial T}{\partial \psi}, \tag{2.18}$$

where n denotes the inward normal co-ordinate. Of particular interest is the average heat flux

$$\bar{Q} \equiv \frac{1}{\lambda}\int_{x=x_0}^{x=x_0+\lambda} Q\, ds = \frac{\kappa}{\lambda}\int_{\phi_0}^{\phi_0+\Delta\phi} \frac{\partial T}{\partial\psi}\frac{d\psi}{dn}\, ds = \frac{\kappa}{\lambda}\int_{\phi_0}^{\phi_0+\Delta\phi} \frac{\partial T}{\partial\psi}\, d\phi, \tag{2.19}$$

$$\bar{Q} = \frac{\kappa}{\lambda}\int_{\phi_0}^{\phi_0+\Delta\phi} \frac{\partial T_0^{(c)}}{\partial\psi}\, d\phi. \tag{2.20}$$

Thus the average heat flux depends only upon $T_0^{(c)}$, and not upon any fluctuating component of the temperature field. This contrasts with results which have been obtained for rotational flow waves, where the average temperature field $T_0^{(c)}(\psi)$ is coupled to the fluctuating components (Omholt 1970). Hence, solutions of (2.14) are not necessary in relating $\bar{Q}$ to $T_0^{(c)}$, and in this paper emphasis will be placed only on the solution to (2.13) for either boundary condition (I) or (II):

$$T_0^{(c)} = T_s + (T_w - T_s)\,\psi/\psi_0 \equiv T_s + \Delta T \psi/\psi_0, \tag{2.21}$$

where T_s is the average surface temperature of the water. If Dirichlet conditions are imposed, then T_s is specified. If $\bar{Q}$ is specified, then from (2.20),

$$\Delta T = \bar{Q}\psi_0 \lambda/\kappa\Delta\phi. \tag{2.22}$$

O'Brien (1967) poses a set of boundary conditions, which in the notation of this paper are that T be periodic in ϕ, and that $T_0^{(c)}$ ($\psi = 0$) and $T_0^{(c)}$ ($\psi = \psi_0$) be specified. No further boundary conditions are stated (although it would seem that, unless they are, the problem is not well-posed). In any case, O'Brien's zeroth-order problem is well-posed, and corresponds to the Dirichlet-boundary-condition version of the problem for $T_0^{(c)}$; O'Brien's results for the Gerstner wave can be compared with the results obtained here.

Following O'Brien, we define an 'equivalent slab' as the fluid within $\phi_0, \phi_0+\Delta\phi$; $0, \psi_0$ reoriented into a rectangular slab of the same width $\Delta x = \lambda$, and having the same thermal boundary conditions. If the heat flux through the surface is held constant, then the temperature difference $\Delta T = T_w - T_s$ decreases as the wave amplitude increases. If ΔT is held constant, the average heat flux $\bar{Q}$ increases with increasing amplitude. A measure of these changes, called here the 'wave effectiveness', is given by

$$W \equiv \left[\frac{(\Delta T)_{\text{equiv. slab}}}{(\Delta T)_{\text{wavy surface}}}\right]_{\bar{Q}\text{ constant}} = \left[\frac{\bar{Q}_{\text{wavy surface}}}{\bar{Q}_{\text{equiv. slab}}}\right]_{\Delta T\text{ constant}} \tag{2.23}$$

The equivalent slab has a horizontal length λ, and a vertical thickness A/λ, A denoting the cross-sectional area of the slab, which is identical to that of the wavy layer. For the equivalent slab, $\bar{Q}$ and ΔT are obviously related by

$$\kappa\Delta T/\bar{Q}(A/\lambda) = 1. \tag{2.24}$$

Using (2.22),

$$W = A\,\Delta\phi/\lambda^2\psi_0. \tag{2.25}$$

The area contained within ϕ_0, $\phi_0+\Delta\phi$; 0, ψ_0 is

$$A = \iint dx\,dy = \int_0^{\psi_0}\int_{\phi_0}^{\phi_0+\Delta\phi} \frac{\partial(x,y)}{\partial(\phi,\psi)}\,d\phi\,d\psi, \tag{2.26}$$

where

$$\left.\begin{aligned} \partial(x,y)/\partial(\phi,\psi) &= (\partial x/\partial\phi)^2+(\partial y/\partial\phi)^2 = |dz/dw|^2,\\ z \equiv x+iy, \quad w &\equiv \phi+i\psi, \quad i \equiv \sqrt{-1}. \end{aligned}\right\} \tag{2.27}$$

Thus

$$W = \frac{\Delta\phi}{\lambda^2\psi_0}\int_0^{\psi_0}\int_{\phi_0}^{\psi_0+\Delta\phi}\left|\frac{dz}{dw}\right|^2 d\phi\,d\psi. \tag{2.28}$$

3. Sinusoidal waves in deep water

The wave field of linear waves in deep water acted upon by gravity and/or surface tension is

$$z = -(w/c)+a\,e^{ikw/c}, \tag{3.1}$$

where a denotes the amplitude of the wave; a wave crest lies at $\phi = \frac{3}{4}\lambda c$ for $+y$ directed downward. The wave-number k and the phase speed c are related by the dispersion relation.

In order to compute the wave effectiveness, one needs

$$\left|\frac{dz}{dw}\right|^2 = \frac{1}{c^2}[1+2ka\sin(k\phi/c)\,e^{-k\psi/c}+k^2a^2\,e^{-2k\psi/c}]. \tag{3.2}$$

By substituting this expression into (2.28) and evaluating the integrals

$$W = 1+k^2a^2\left[\frac{c}{2k\psi_0}(1-e^{-2k\psi_0/c})\right]. \tag{3.3}$$

The argument of the exponential in (3.3) is usually tiny, because $\psi_0/c \simeq$ 1–2 mm, the depth of the thermal boundary layer. Hence, $k\psi_0/c \ll 1$ for all but small-wavelength capillary waves. When $k\psi_0/c \ll 1$,

$$W = 1+k^2a^2. \tag{3.4}$$

In (3.4) one may distinguish three contributions which waves make to $(W-1)$. First, in one wavelength, the length of free surface exceeds λ by a factor $1+\frac{1}{4}k^2a^2$. Second, the area-preserving requirement implies a corresponding decrease of the mean layer thickness by the same factor, $1+\frac{1}{4}k^2a^2$. Third, the thickness of the layer varies by a factor $1-ka\sin(k\phi/c)$ from its mean thickness; this contributes a factor of $1+\frac{1}{2}k^2a^2$ to W because of the following. The fluid at ψ_0 is physically closer to the free surface at a wave trough than the mean thickness by the factor $(1-ka)$ and the fluid at a crest is further by the factor $(1+ka)$. For a given ΔT, the excess of heat transported at trough regions exceeds the deficit transported at a crest region by the factor

$$\langle(1-ka\sin k\phi/c)^{-1}\rangle = 1+\tfrac{1}{2}k^2a^2.$$

4. Capillary waves of arbitrary amplitude

Crapper (1957) has derived an exact solution for two-dimensional capillary waves in an incompressible inviscid irrotational fluid. His solution is in perfect form for (2.28). In particular, his equation (56) becomes

$$c\frac{dz}{dw} = \left(\frac{1-A\,e^{ikw/c}}{1+A\,e^{ikw/c}}\right)^2, \tag{4.1}$$

where A is an amplitude parameter. Defining a as $\frac{1}{2}$ the vertical distance between crest and trough (this differs from Crapper's a):

$$A = (2/ka)\left[(1+\tfrac{1}{4}k^2a^2)^{\frac{1}{2}}-1\right] \quad (0 < ka < 2{\cdot}29). \tag{4.2}$$

Crapper shows that waves exist up to a limiting amplitude of $ka = 2{\cdot}29$, at which point the wave profiles are cuspated, and beyond which they presumably break.

Equation (4.1) is of such a form that the integrations indicated in (2.28) can be executed analytically. The algebra is lengthy, and is placed in appendix A. The result is

$$W = 1+\frac{2c}{k\psi_0}\left[\left(\frac{1+A^2}{1-A^2}\right)^2-\left(\frac{1+A^2e^{-2k\psi_0/c}}{1-A^2e^{-2k\psi_0/c}}\right)^2\right]. \tag{4.3}$$

In the limit $k\psi_0/c \to 0$, the case of infinitesimal boundary-layer thickness,

$$W = 1+\frac{16A^2(1+A^2)}{(1-A^2)^3} = 1+k^2a^2+\tfrac{1}{8}k^4a^4+\dots. \tag{4.4}$$

In figure 1 W is plotted as a function of amplitude for various values of $2k\psi_0/c$, using equations (4.2) and (4.3) or (4.4). The capillary wave of maximum amplitude corresponds to $ka = 2{\cdot}29$, where each curve terminates. The values of W for capillary waves can range all the way up to $9{\cdot}0$, which indicates that capillary waves have the potential of being very effective in modifying the thermal boundary-layer structure.

The capillary-wave theory of Crapper (1957) ignores gravity. Linear water waves with gravity are qualitatively capillary only when wavelengths are smaller than 1·7 cm, and so Crapper's theory can be presumed to be applicable

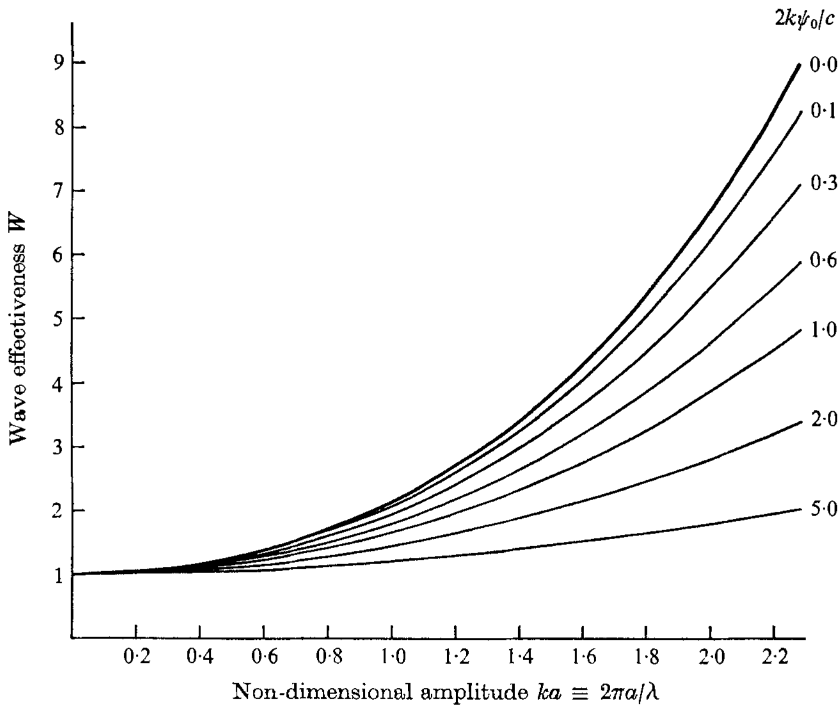

FIGURE 1. Wave effectiveness of capillary waves. When $2k\psi_0/c \to 0$ the thermal boundary layer is negligibly thin when compared to the wavelength; when $2k\psi_0/c \to \infty$, the layer is infinitely larger than the wavelength.

only for $\lambda < 1{\cdot}7$ cm and accurate only for $\lambda \gtrsim 1$ cm. Thus the condition $k\psi_0/c \ll 1$ is seldom a good approximation for thermal boundary layers having thicknesses $\sim$ 1–2 mm. In figure 2 two fixed values of A/λ, the thickness of the equivalent slab, are assumed (1 mm and 2 mm). The wave effectiveness is plotted against wavelength for the capillary wave of limiting amplitude, in order to show the range of capillary waves which are capable of providing large thermal effects. This range is not negligible.

5. Stokes waves

Stokes (1847) was the first to deduce the approximate structure of finite-amplitude gravity waves in deep water. The solution involves an expansion in powers of ka. Stokes (1880) recast the theory into a form which is followed here. The flow field is governed by

$$z = -\frac{w}{c} - \frac{i}{k} \sum_{n=1}^{\infty} A_n e^{inkw/c}. \tag{5.1}$$

Certain signs in (5.1) differ from those written by Stokes, because his $+y$ direction was upward, and his ψ was defined as $+\int v\,ds = -\int u\,dy$. Stokes computes the first five values of A_n in terms of a parameter $b \equiv A_1$; Wilton (1914) computes A_n through A_{12}. The series probably converges only up to $(ka) \simeq \frac{1}{3}$ (Wilton 1914).

Because the wavelength of a pure gravity wave is always vastly greater than 1–2 mm, the wave effectiveness is evaluated only in the limit of negligibly thin boundary layers, i.e. $k\psi_0/c \to 0$. Then, (2.28) can be re-written

$$W = \frac{\Delta\phi}{\lambda^2}\int_{\phi_0}^{\phi_0+\Delta\phi}\left|\frac{dz(\phi, 0)}{dw}\right|^2 d\phi. \tag{5.2}$$

After much algebra, which is sketched in appendix B, and using only the first five A_n:

$$W = 1+(ka)^2+7(ka)^4+\tfrac{227}{4}(ka)^6+O(ka)^8. \tag{5.3}$$

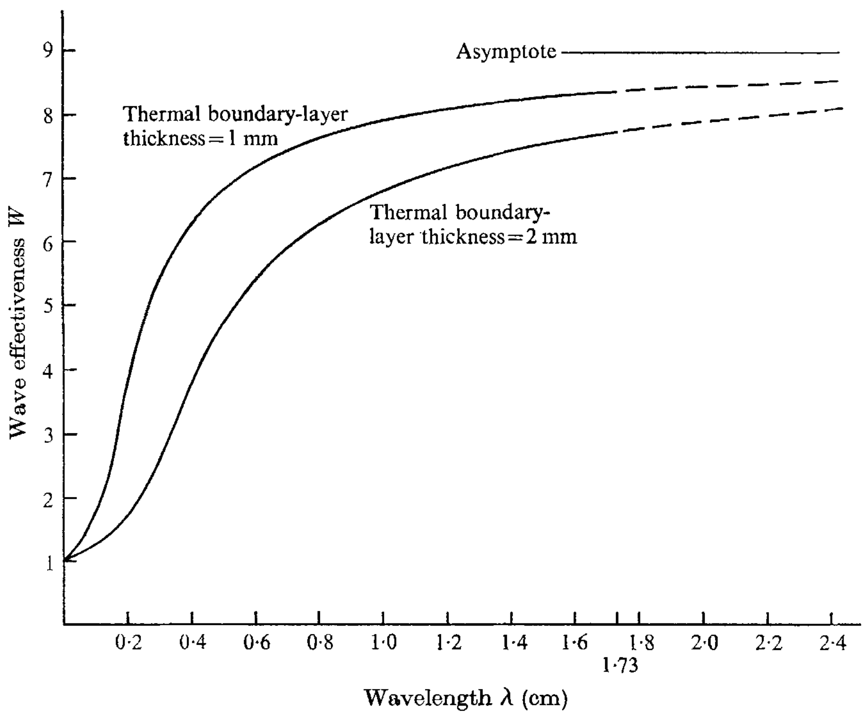

FIGURE 2. Wave effectiveness of capillary waves of maximum amplitude. The asymptote is the value of W for $k\psi_0/c \to 0$. The wavelength 1·73 cm corresponds to the slowest linear gravity-capillary wave in deep water.

Presumably, (5.3) converges for sufficiently small values of ka. It probably does not converge for values of ka near breaking. The limiting value of the ratio of wave height to length is 0·1418 (Havelock 1918), which corresponds to $ka = 0{\cdot}4461$. Inserting this value into (5.3), we get

$$W = 1+0{\cdot}1990+0{\cdot}2772+0{\cdot}4472+\ldots. \tag{5.4}$$

The contribution from each term of the series beyond the second is larger than the preceding one, indicating that equation (5.3) is not particularly illuminating for large-amplitude Stokes waves.

6. The gravity wave of limiting amplitude

Michell (1893) computed the structure of the deep-water irrotational gravity wave of maximum amplitude. Havelock (1918) extended his result to large-amplitude waves, including the breaking wave, in a slightly different formulation. Here we follow Havelock's treatment.

The wave structure is governed by:

$$dz/dw = 2^{-\frac{1}{3}}c^{-1}(-i\sin\tfrac{1}{2}kw/c)^{-\frac{1}{3}}\,e^{-ikw/6c}[1+c_1e^{ikw/c}+c_2e^{2ikw/c}+\ldots] \quad (6.1)$$

or by

$$dw/dz = 2^{\frac{1}{3}}c(+i\sin\tfrac{1}{2}kw/c)^{+\frac{1}{3}}\,e^{+ikw/6c}[1+b_1e^{ikw/c}+b_2e^{2ikw/c}+\ldots]. \quad (6.2)$$

Havelock computes the coefficients b_n up to b_4 for the wave of maximum amplitude. By substituting (6.1) into (5.2), relating the c_n's to the b_n's and carrying out the integrations (see appendix C)

$$W = 1\cdot461-0\cdot065-0\cdot011-0\cdot003-0\cdot001 = 1\cdot381\mp0\cdot002. \quad (6.3)$$

Basically, (see appendix C for the precise details) the first term of (6.3) comes from the first term of (6.1)–(6.2), the first two terms of (6.3) come from the first two terms of equations (6.1)–(6.2) etc. Thus, most of the contribution to $(W-1)$ in the presence of the wave of limiting amplitude lies in the lowest order; the numerical values of each successive order decrease fast enough that W must lie quite close to 1·381.

O'Brien (1967) has obtained a value $W = 2\cdot00$ for a negligibly thin thermal boundary layer on the Gerstner wave of limiting amplitude. The overall reason that the irrotational gravity wave is so much less effective than the Gerstner wave in modifying thermal boundary layers is that it breaks at a much smaller amplitude, $ka = 0\cdot4461$, rather than $ka = 1\cdot0000$. Consequently, the total surface length per wavelength s/λ, and the shrinking of the mean thermal boundary-layer thickness are substantially less for the limiting irrotational gravity wave ($s = 1\cdot037\lambda$; see appendix C) than for the limiting Gerstner wave ($s = 2\frac{1}{2}\lambda$).

7. Concluding remarks

7.1. *Comparison of the results for various wave types*

Table 1 summarizes the results of these calculations and an expansion of equation (3.16) of O'Brien (1967). The expressions agree to $O(ka)^2$. There is no reason to expect agreement beyond this order, and there is none. The coefficients of the $(ka)^4$ term happen to be small for the Gerstner wave and the capillary wave, and large for the gravity wave. This is undoubtedly related to the exact geometry of the waves. The significant differences in maximum wave effectiveness for the three types of waves are related most directly to the different maximum values of (ka) which these waves can attain.

7.2. *Relevance of the results*

The parameter W is a reasonable estimate of the factor by which the temperature difference ΔT across a thermal boundary layer decreases, when the average heat flux out of water is relatively independent of the presence of the waves.† The computed values of W show that capillary waves can decrease ΔT by a factor which is nearly W, which theoretically can be as large as 9·0 for infinitesimal layers, and about 7–8·5 for 1–2 mm layers. Irrotational non-breaking gravity waves are not nearly as effective, because W cannot exceed 1·38.

In reality, because of viscous damping and instabilities arising from resonant interaction among wave trains (McGoldrick 1965) one does not expect to see long trains of capillary waves at near-breaking amplitudes. Nevertheless, if even small patches of ocean surface contain large-amplitude transient capillary waves, the modifications to the thermal boundary layers by the capillary waves may exceed those of the gravity waves; in no event should the capillary waves be ignored.

Wave type		Wave effectiveness W	Maximum ka	Maximum W
Capillary		$1+(ka)^2+\frac{1}{8}(ka)^4+O(ka)^6$	2·29	9·0
Gravity	Stokes	$1+(ka)^2+7(ka)^4+O(ka)^6$	—	—
(irrotational)	Limiting	—	0·45	1·38
Gerstner (rotational grav.)		$1+(ka)^2$	1·00	2·00

TABLE 1. Summary of results. The wave amplitude is a; the wave-number is k. The Stokes expansion of gravity waves fails to converge at values of ka much less than that appropriate for the wave of limiting amplitude.

The theoretical results presented here may constitute an explanation of the results of laboratory experiments recently conducted by Hill (1970). He measured ΔT with $\bar{Q}$ held constant in a short-fetch wind-wave tank. He found an almost discontinuous drop in ΔT by a factor of approximately 3 with increasing wind speed or fetch. In all cases the drop occurred when waves, mostly capillary, became visible. No measurements of wave properties were made.

The marked drop of ΔT may simply have been a manifestation of the large decrease in temperature gradients demanded by the heat conduction equation when large-amplitude capillary waves are present. Results of this theory, as summarized in figures 1–2, show that there is a fairly-wide range of wavelengths

† If radiation and evaporation dominate surface cooling, it is reasonable to assume that the average heat flux with waves usually exceeds that without waves by the factor s/λ, the surface length per wavelength. This factor is $1+\frac{1}{4}(ka)^2$ for linear waves ($W = 1+(ka)^2$), and, as shown in appendix C, is 1·037 for the gravity wave of limiting amplitude ($W = 1{\cdot}38$). For capillary waves of large amplitude there is a 'blocking effect' possibly involving some trapped water–vapour and certainly some absorption of radiation emitted by one part of the wave at another part. Thus s/λ will overestimate the changes in heat flux. Let s' denote the length of free surface over one wavelength over which an outward normal does not intercept another part of the wave. Then the factor by which waves change $\bar{Q}$ is approximately s'/λ. For the capillary wave of limiting amplitude, $s'/\lambda = 1{\cdot}04$. In all of these cases, $(\bar{Q}_{\text{wavy}}/\bar{Q}_{\text{still}} - 1) < \frac{1}{4}(W-1)$.

and amplitudes over which W exceeds 3. Measurements of the wave properties would be necessary in order to make a closer comparison between theory and experiment; they are now being planned.

I wish to thank Dr Steve A. Piacsek of Argonne National Laboratory and Prof. George W. Platzman of the University of Chicago for useful discussions related to this problem, and an anonymous referee who brought the papers by Boussinesq and Grosh & Cess to my attention.

Appendix A. Evaluation of the wave effectiveness for capillary waves

The details of the computations involved in evaluating W for capillary waves are laid out here. The starting point is the Jacobian following (4.1)

$$\left|\frac{dz}{dw}\right|^2 = \frac{1}{c^2}\left[\frac{(1-A\,e^{ikw/c})\,(1-A\,e^{-ikw^*/c})}{(1+A\,e^{ikw/c})\,(1+A\,e^{-ikw^*/c})}\right] \tag{A 1}$$

$$= \frac{1}{c^2}\left(\frac{1-2A\cos(k\phi/c)\,e^{-k\psi/c}+A^2\,e^{-2k\psi/c}}{1+2A\cos(k\phi/c)\,e^{-k\psi/c}+A^2\,e^{-2k\psi/c}}\right)^2, \tag{A 2}$$

where w^* is the complex conjugate of w.

Let ξ denote $k\phi/c$ and μ denote $k\psi/c$, μ_0 denoting $k\psi_0/c$. With $\Delta\phi = \lambda c$, (2.28) can be rewritten in terms of dimensionless variables

$$W = \frac{c^2}{2\pi\mu_0}\int_0^{\mu_0} d\mu \int_0^{2\pi} d\xi \left|\frac{dz}{dw}\right|^2 \tag{A 3}$$

$$= \frac{1}{2\pi\mu_0}\int_0^{\mu_0} d\mu \int_{-\pi}^{\pi} d\xi \left[\frac{1-B\cos\xi}{1+B\cos\xi}\right]^2 \equiv \frac{1}{2\pi\mu_0}\int_0^{\mu_0} I\,d\mu, \tag{A 4}$$

where

$$B = 2A\,e^{-\mu}/(1+A^2 e^{-2\mu}). \tag{A 5}$$

First we consider the integral I,

$$I \equiv \int_{-\pi}^{\pi} d\xi \left[\frac{1-B\cos\xi}{1+B\cos\xi}\right]^2 = 2\int_0^{\pi} d\xi \left[\frac{1-B\cos\xi}{1+B\cos\xi}\right]^2. \tag{A 6}$$

Noting that the integrand of (A 6) can be written

$$\left[\frac{1-B\cos\xi}{1+B\cos\xi}\right]^2 = 1 - \frac{4}{1+B\cos\xi} + \frac{4}{(1+B\cos\xi)^2}, \tag{A 7}$$

then (A 6) can be represented as the sum of three integrals, each of which is identical in form to § 3.645 of Gradshteyn & Ryzhik (1965). Finally

$$I = 2\pi[1-4(1-B^2)^{-\frac{1}{2}}+4(1-B^2)^{-\frac{3}{2}}]. \tag{A 8}$$

Now

$$(1-B^2)^{-\frac{1}{2}} = \left[1-\frac{4A^2 e^{-2\mu}}{(1+A^2 e^{-2\mu})^2}\right]^{-\frac{1}{2}} = \frac{1+A^2 e^{-2\mu}}{1-A^2 e^{-2\mu}}. \tag{A 9}$$

If a parameter α is defined by $A^2 \equiv e^{-2\alpha}$, then

$$(1-B^2)^{-\frac{1}{2}} = \coth(\mu+\alpha) \tag{A 10}$$

and, using (A 10) and (A 8), (A 4) becomes

$$W = \frac{1}{\mu_0}\int_0^{\mu_0} [1-4\coth(\alpha+\mu)+4\coth^3(\alpha+\mu)]\,d\mu. \tag{A 11}$$

The indefinite integrals corresponding to (A 11) are in Gradshteyn & Ryzhik (1965, §2.42) and

$$W = 1+\frac{2}{\mu_0}\{\coth^2\alpha-\coth^2(\mu+\alpha)\} \tag{A 12}$$

$$= 1+\frac{2}{\mu_0}\left[\left(\frac{1+A^2}{1-A^2}\right)^2-\left(\frac{1+A^2e^{-2\mu_0}}{1-A^2e^{-2\mu_0}}\right)^2\right], \tag{A 13}$$

which is equivalent to (4.3).

Appendix B. Evaluation of the wave effectiveness for Stokes waves

The details of the computations involved in evaluating W for Stokes waves are laid out here. The starting point is (5.2), which in the notation of appendix A is

$$W = \frac{c^2}{2\pi}\int_{-\pi}^{\pi}\left|\frac{dz(\phi,0)}{dw}\right|^2 d\xi. \tag{B 1}$$

It is convenient to use

$$\left|\frac{dz}{dw}\right|^2 = \left(\frac{\partial x}{\partial\phi}\right)^2+\left(\frac{\partial y}{\partial\phi}\right)^2, \tag{B 2}$$

where

$$\left(\frac{\partial x}{\partial\phi}\right)^2 = \frac{1}{c^2}\left[1-\sum_{n=1}^{\infty} n\,A_n\cos(n\xi)\right]^2;\ \left(\frac{\partial y}{\partial\phi}\right)^2 = \frac{1}{c^2}\left[\sum_{n=1}^{\infty} n\,A_n\sin(n\xi)\right]^2. \tag{B 3}$$

Because

$$\int_{-\pi}^{\pi}\sin(n\xi)\sin(m\xi)\,d\xi = \int_{-\pi}^{\pi}\cos(n\xi)\cos(m\xi)\,d\xi = \begin{Bmatrix}\pi;\ m=n\\ 0;\ m\neq n\end{Bmatrix}, \tag{B 4}$$

it follows from direct substitution of (B 3) via (B 2) into (B 1) that

$$W = 1+\sum_{n=1}^{\infty} n^2A_n^2. \tag{B 5}$$

Stokes (1880) computed the first few A_n. Defining $b \equiv A_1$,

$$A_1 \equiv b;\ A_2 = -(b^2+\tfrac{1}{2}b^4);\quad A_3 = -(\tfrac{3}{2}b^3+\tfrac{19}{12}b^5);\quad A_4 = -\tfrac{8}{3}b^4;\ A_5 = \tfrac{125}{24}b^5. \tag{B 6}$$

If a is defined in a way consistent with that defined for capillary waves, i.e. one-half the vertical distance between crest and trough, the crest here being at $k\phi/c = 0$ and the trough at $k\phi/c = \pi$,

$$a = \tfrac{1}{2}[y(\xi=0)-y(\xi=\pi)], \tag{B 7}$$

$$ka = A_1+A_3+A_5+\ldots = b-\tfrac{3}{2}b^3+\tfrac{29}{8}b^5+O(b^7). \tag{B 8}$$

Using only those A_n up to A_5, $(ka)^2$ can be determined only to an accuracy of b^6. To this accuracy, (B 5) with (B 6) yields

$$W = 1+b^2+4b^4+\tfrac{97}{4}b^6. \tag{B 9}$$

With the help of (B 8) this result can be rewritten as a series in powers of k^2a^2:

$$W = 1+k^2a^2+7k^4a^4+(227/4)\,k^6a^6. \tag{5.3}$$

Appendix C. Evaluation of the wave effectiveness and surface length of the gravity wave of maximum amplitude

The details of the computations involved in evaluating W and s/λ for the gravity wave of maximum amplitude are laid out here. The starting point is Havelock's expansion (6.1). Using the notation of appendix A, along $\psi = 0$

$$\left|\frac{dz(\phi,0)}{dw}\right|^2 = \frac{1}{2^{\frac{2}{3}}c^2}(\sin\tfrac{1}{2}\xi)^{-\frac{2}{3}}\left[\left(1+\sum_{n=1}^{\infty} c_n\cos(n\xi)\right)^2+\left(\sum_{n=1}^{\infty} c_n\sin(n\xi)\right)^2\right]. \quad \text{(C 1)}$$

By substituting this expression into (5.2)

$$W = \frac{1}{2^{\frac{5}{3}}\pi}\int_0^{2\pi}(\sin\tfrac{1}{2}\xi)^{-\frac{2}{3}}\left[\left(1+\sum_{n=1}^{\infty} c_n\cos(n\xi)\right)^2+\left(\sum_{n=1}^{\infty} c_n\sin(n\xi)\right)^2\right]d\xi \quad \text{(C 2)}$$

$$= \frac{1}{2^{\frac{2}{3}}\pi}\int_0^{\pi}(\sin\xi')^{-\frac{2}{3}}\left[\left(1+\sum_{n=1}^{\infty} c_n\cos(2n\xi')\right)^2+\left(\sum_{n=1}^{\infty} c_n\sin(2n\xi')\right)^2\right]d\xi'. \quad \text{(C 3)}$$

All products $\cos(2p\xi')\cos(2q\xi')$ and $\sin(2p\xi')\sin(2q\xi')$, which occur within the series in the integrand of (C 3), involve factors $\cos(2m\xi')$. Collecting the coefficients of like terms, and retaining only terms to order 4, the expression within the square parentheses of (C 3) is

$$[\ \ ] = (1+c_1^2+c_2^2)+(2c_1+2c_1c_2)\cos(2\xi')+(2c_2+2c_1c_3)\cos(4\xi')$$
$$+2c_3\cos(6\xi')+2c_4\cos(8\xi'). \quad \text{(C 4)}$$

The wave effectiveness is thus of the form

$$W = C_0I_0+C_1I_1+\ldots+C_4I_4+\ldots, \quad \text{(C 5)}$$

where $C_0 = 1+c_1^2+c_2^2$; $C_1 = 2c_1+2c_1c_2$, etc. and

$$I_m = \frac{1}{2^{\frac{2}{3}}\pi}\int_0^{\pi}\sin^{-\frac{2}{3}}\xi'\cos(2m\xi')\,d\xi' \quad \text{(C 6)}$$

$$= (-)^m\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3}+m)\,\Gamma(\frac{2}{3}-m)}, \quad \text{(C 7)}$$

where Γ is the gamma function; proceeding from (C 6) to (C 7) requires the 8th integral of 3·631 of Gradshteyn & Ryzhik (1965).

Using the fact that $\Gamma(z+1) = z\,\Gamma(z)$ it is easy to show that

$$I_m = \frac{3m-2}{3m-1}I_{m-1}. \quad \text{(C 8)}$$

Because $I_0 = 1{\cdot}459$, it follows that $I_1 = 0{\cdot}729$, $I_2 = 0{\cdot}5825$, etc.

Havelock gave the following values for the coefficients b_n in (6.2):

$$b_1 = 0{\cdot}0414; \quad b_2 = 0{\cdot}0114; \quad b_3 = 0{\cdot}0042; \quad b_4 = 0{\cdot}0014.$$

It thus follows that

$$c_1 = -b_1 = -0{\cdot}0414,$$
$$c_2 = b_1^2-b_2 = -0{\cdot}0097,$$
$$c_3 = 2b_1b_2-b_3-b_1^3 = -0{\cdot}0032,$$
$$c_4 = 2b_1b_3-3b_1^2b_2-b_4+b_2^2+b_1^4 = -0{\cdot}0010.$$

The value of W can be evaluated from (C 5). The numbers within the first parentheses of each term of the following equation form C_n, the numbers following the order in (C 4):

$$W = (1\cdot0000+0\cdot0017+0\cdot0001)(1\cdot459) + (-0\cdot0828+0\cdot0008)(0\cdot729)+(-0\cdot0194+0\cdot0003)(0\cdot5825) + (-0\cdot0064)(0\cdot510)+(-0\cdot0020)(0\cdot463). \tag{C 9}$$

$$W = 1\cdot461-0\cdot065-0\cdot011-0\cdot003-0\cdot001 = 1\cdot381. \tag{6.3}$$

The surface length per unit wavelength for the gravity wave of maximum amplitude is

$$\int_{1\lambda} ds = \int_{\phi_0}^{\phi_0+\Delta\phi} \frac{\partial s}{\partial \phi} d\phi = \int_{\phi_0}^{\phi_0+\Delta\phi} [(\partial x/\partial\phi)^2+(\partial y/\partial\phi)^2]^{\frac{1}{2}} d\phi \tag{C 10}$$

$$= \int_{\phi_0}^{\phi_0+\Delta\phi} \left|\frac{dz}{dw}\right| d\phi. \tag{C 11}$$

By substituting the square root of (C 1) into (C 11):

$$s/\lambda = \frac{1}{2^{\frac{1}{3}}\pi}\int_0^\pi (\sin\xi')^{-\frac{1}{3}} \left[\left(1+\sum_{n=1}^{\infty} c_n \cos(2n\xi')\right)^2 + \left(\sum_{n=1}^{\infty} c_n \sin(2n\xi')\right)^2\right]^{\frac{1}{2}} d\xi' \tag{C 12}$$

$$= \frac{1}{2^{\frac{1}{3}}\pi}\int_0^\pi (\sin\xi')^{-\frac{1}{3}} \left[\sum_{n=1}^{\infty} d_n \cos(2n\xi')\right] d\xi', \tag{C 13}$$

where d_n can be obtained from c_n and, therefore, from b_n. The integrals in (C 13) are practically of the same form as (C 6) and are included in Gradshteyn & Ryzhik's 8th integral of 3·631. After much algebra,

$$s/\lambda = (1\cdot001)[1\cdot063-0\cdot022-0\cdot004-0\cdot001-0\cdot000] = 1\cdot037. \tag{C 14}$$

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